# A SECTIONAL CURVATURE FOR STATISTICAL STRUCTURES

BARBARA OPOZDA

Instytut Matematyki UJ, ul. Łojasiewicza 6, 30-348 Cracow, Poland
Barbara.Opozda@im.uj.edu.pl

Abstract. A new type of sectional curvature is introduced. The notion is purely algebraic and can be located in linear algebra as well as in differential geometry.

## 1. Introduction

Sectional curvature is one of the most important concepts in differential geometry. Nevertheless, it is attributed to Riemannian or pseudo-Riemannian geometry only. The curvature tensor field is defined for any connection but to define a sectional curvature, which assigns to a vector plane of a tangent space a number, seems to need a scalar product. Moreover, the metric and the connection must be related in a good manner. For instance, in the classical affine differential geometry one has a metric tensor field and the so called induced connection, but the curvature tensor of type $(0,4)$ constructed by these objects does not have enough symmetries. The tensor satisfies appropriate symmetry conditions for affine spheres but it leads to trivial cases, namely to spaces of constant sectional curvature. The problem can be solved by adding to the curvature tensor the curvature tensor for the dual connection. This idea is discussed in [8] for statistical structures on abstract manifolds, that is, on manifolds (not necessarily immersed in any standard space) endowed with a matric tensor field $g$ and a torsion-free affine connection $\nabla$ for which $\nabla g$ as a 3-covariant tensor is symmetric.

A statistical structure is also called a Codazzi structure, see e.g. [7], [6]. We use the name "statistical structure" following [5] or [3]. The name "Codazzi structure" may refer to all situations, where we have any tensor field whose covariant derivative is totally symmetric.

The geometry of affine hypersurfaces in the standard affine space $\mathbf{R}^n$ or, more generally, the geometries of the second fundamental form, including the theory of Lagrangian submanifolds in complex space forms, are natural sources of statistical structures. However, the fact that the structures are induced by the simple structures on the ambient spaces imposes strong conditions on the induced statistical structure. For instance, for affine hypersurfaces, it it necessary that the dual connection is projectively flat.

It turns out that for statistical structures one can define few sectional curvatures. In [8] we studied the sectional $\nabla$-curvature, that is, a sectional curvature determined by a metric tensor and a connection $\nabla$. In this paper we propose another type of sectional curvature. Its idea is purely algebraic. This sectional curvature can be

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defined on any vector space endowed with a scalar product and a symmetric cubic form. Then it can be transfered to statistical structures on manifolds. In this paper we provide some basic information on this sectional curvature and we give exemplary theorems concerning this notion.

## 2. Statistical structures

One can define a statistical structure on a manifold $M$ in three equivalent ways. First of all $M$ must have a Riemannian structure defined by a metric tensor field $g$. Throughout the paper we assume that $g$ is positive definite, although $g$ can be also indefnite. A statistical structure can be defined as a pair $(g, K)$ on a manifold $M$, where $g$ is a Riemannian metric tensor field and $K$ is a symmetric $(1,2)$-tensor field which is also symmetric relative to $g$, that is, the cubic form

$$C(X, Y, Z) = g(X, K(Y, Z)) \tag{1}$$

is symmetric relative to $X, Y$. It is clear that any symmetric cubic form $C$ on a Riemannian manifold $(M, g)$ defines by (1) a $(1,2)$-tensor field $K$ having the symmetry properties as above. Another equivalent definition says that a statistical structure is a pair $(g, \nabla)$, where $\nabla$ is a torsion-free affine connection on $M$ and $\nabla g$ as a $(0,3)$-tensor field on $M$ is symmetric in all arguments. Let us fix that for a tensor field $s$ and a connection $\nabla$ the notation $\nabla s(X, ...)$ stands for $(\nabla_X s)(...)$. The affine connection $\nabla$ from the last definition equals to $\hat{\nabla} + K$, where $\hat{\nabla}$ is the Levi-Civita connection for $g$ and $K$ is the difference tensor. Since $\nabla g(X, Y, Z)) = -2g(K(X, Y), Z)$, we obtain a statistical structure $(g, K)$ from $(g, \nabla)$. We shall call $\nabla$ a statistical connection. A manifold equipped with a statistical structure will be called a statistical manifold.

For any connection $\nabla$ on a Riemannian manifold $(M, g)$ one defines its conjugate connection $\overline{\nabla}$ (relative to $g$) as follows

$$g(\nabla_X Y, Z) + g(Y, \overline{\nabla}_X Z) = Xg(Y, Z) \tag{2}$$

for any vector fields $X, Y, Z$ on $M$. The connections $\nabla$ and $\overline{\nabla}$ are simultaneously torsion-free. It is also known that if $(g, \nabla)$ is a statistical structure then so is $(g, \overline{\nabla})$. Moreover, if $(g, \nabla)$ is trace-free then so is $(g, \overline{\nabla})$, see e.g. [6]. Recall that a trace-free statistical structure is such a structure for which $\operatorname{tr}_g(\nabla g)(X, \cdot, \cdot) = 0$ for every $X$ or equivalently $\operatorname{tr}_g K = 0$, or equivalently $\operatorname{tr} K_X = 0$ for every $X$, where $K_X Y = K(X, Y)$. Note that a statistical structure is trace-free if and only if $\nabla \nu_g = 0$, where $\nu_g$ is the volume form determined by $g$. If $R$ is the curvature tensor for $\nabla$ and $\overline{R}$ is the curvature tensor for $\overline{\nabla}$ then we have, [6],

$$g(R(X, Y)Z, W) = -g(\overline{R}(X, Y)W, Z) \tag{3}$$

for every $X, Y, Z, W$. In particular, $R = 0$ if and only if $\overline{R} = 0$. If $K$ is the difference tensor between $\nabla$ and $\hat{\nabla}$, that is,

$$\nabla_X Y = \hat{\nabla}_X Y + K_X Y, \tag{4}$$

then

$$\overline{\nabla}_X Y = \hat{\nabla}_X Y - K_X Y. \tag{5}$$

It is also known that

$$R(X, Y) = \hat{R}(X, Y) + (\hat{\nabla}_X K)_Y - (\hat{\nabla}_Y K)_X + [K_X, K_Y]. \tag{6}$$

Writing the same equality for $\overline{\nabla}$ and adding both equalities we get

$$R(X,Y) + \overline{R}(X,Y) = 2\hat{R}(X,Y) + 2[K_X, K_Y]. \tag{7}$$

The following lemma follows from formulas (3), (6) and (7).

**Lemma 2.1.** *Let $(g, K)$ be a statistical structure. The following conditions are equivalent:*
1) $R = \overline{R}$,
2) $\hat{\nabla} K$ *is symmetric,*
3) $g(R(X,Y)Z, W)$ *is skew-symmetric for* $Z, W$.

A statistical structure is called Hessian if the connection $\nabla$ is flat, that is, $R = 0$. In this case, by (7), we have

$$\hat{R} = -[K, K]. \tag{8}$$

For a statistical structure one defines the vector field $E$ by

$$E = \operatorname{tr}_g K. \tag{9}$$

If $e_1, ..., e_n$ is an orthonormal basis of $\mathcal{V}$ then

$$E = (\operatorname{tr} K_{e_1})e_1 + ... + (\operatorname{tr} K_{e_n})e_n. \tag{10}$$

For more information on dual connections, affine differential geometry and statistical structures we refer to [7], [4], [6], [3], [5], [9], [8].

## 3. The sectional $K$-curvature

First we shall give an algebraic setting of the sectional $K$-curvature. Let $\mathcal{V}$ be a vector space with a positive definite scalar product $g$. Let $K$ be a symmetric tensor field of type $(1,2)$ on $\mathcal{V}$ and symmetric relative to $g$. Hence $K_X$ is a tensor of type $(1,1)$ symmetric relative to $g$. In particular, it is diagonalizable. $K$ defines a symmetric cubic form $C$ given by (1).

The tensor field $K$ determines a $(1,3)$-tensor $[K,K]$ given by

$$[K,K](X,Y)Z := [K_X, K_Y]Z = K_X K_Y Z - K_Y K_X Z.$$

This is a curvature-like tensor, that is, it satisfies the following conditions

$$\begin{aligned}
&[K,K](X,Y) = -[K,K](Y,X)\\
&[K,K](X,Y)Z + [K,K](Y,Z)X + [K,K](Z,X)Y = 0\\
&g([K,K](X,Y)Z, W) = -g([K,K](X,Y)W, Z)
\end{aligned}$$

for every vectors $X, Y, Z, W \in \mathcal{V}$. It follows that we can define the sectional $K$-curvature by a vector plane $\pi$ in $\mathcal{V}$ as follows. Take an orthonormal basis $X, Y$ of $\pi$ and set

$$k(\pi) = g([K,K](X,Y)Y, X). \tag{11}$$

The number $k(\pi)$ is independent of the choice of an orthonormal basis $X, Y$. The sectional $K$-curvature by a plane spanned by vectors $X, Y$ will be denoted by $k(X \wedge Y)$.

On a 2-dimensional vector space $\mathcal{V}$ we have $[K,K](X,Y)Z = k(\mathcal{V})[g(Y,Z)X - g(X,Z)Y]$ for all vectors $X, Y, Z \in \mathcal{V}$. If the dimension of $\mathcal{V}$ is arbitrary and the

sectional $K$-curvature is equal to some constant number $A$ for all vector planes in $\mathcal{V}$ then we have

$$[K,K](X,Y)Z = A[g(Y,Z)X - g(X,Z)Y] \tag{12}$$

for every $X,Y,Z \in \mathcal{V}$. The condition (12) can be written equivalently as

$$\begin{aligned} g(K(X,W),K(Y,Z)) - g(K(Y,W),K(X,Z)) \\ = A[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)] \end{aligned} \tag{13}$$

for every $X,Y,Z,W \in \mathcal{V}$.

The sectional $K$-curvature can be now introduced on a statistical manifold $(M,g,K)$ in the above manner on each tangent space. In general, Schur's lemma does not hold. It follows from the fact that the curvature tensor $[K,K]$, in general, does not satisfy any second Bianchi identity. The following identity can be regarded as the second Bianchi identity for the curvature tensor $R+\overline{R}$, see [8],

**Lemma 3.1.** *For any statistical structure $(g,\nabla)$ we have*

$$\Xi_{U,X,Y}(\hat{\nabla}_U(R+\overline{R}))(X,Y) = \Xi_{U,X,Y}(K_U\cdot(\overline{R}-R))(X,Y),$$

*where $\Xi$ stands for the cyclic permutation sum.*

Using Lemmas (2.1 and (3.1) one easily gets the following analogue of Schur's lemma

**Proposition 3.2.** *Let $(g,K)$ be a statistical structure on a connected manifold $M$ whose dimension is greater than 2. If the (1,3)-tensor field $\hat{\nabla}K$ is symmetric and the sectional $K$-curvature depends only on a point of $M$ then the sectional $K$-curvature is constant on $M$.*

**Example 3.3.** Let $e_1,...,e_n$ be an orthonormal frame of $\mathcal{V}$. Define a $(1,2)$-tensor $K$ on $\mathcal{V}$ as follows

$$K(e_1,e_1) = \lambda e_1, \quad K(e_1,e_i) = \frac{\lambda}{2}e_i, \quad K(e_i,e_i) = \frac{\lambda}{2}e_1, \quad K(e_i,e_j) = 0 \tag{14}$$

for $i,j \geq 2$, $i \neq j$. By a straightforward computation one can check that the sectional $K$-curvature is constant on $\mathcal{V}$ and equals to $\lambda^2/4$. Observe that in this case the cubic form $C$ vanishes on the $(n-1)$-dimensional hyperplane spanned by $e_2,...,e_n$ and the sectional curvature is positive for all sections.

**Example 3.4.** In [1] B-Y Chen studied Lagrangian $H$-umbilical submanifolds. That study leads to the following examples of statistical structures. Let $g$ be a scalar product on a vector space $\mathcal{V}$. In some orthonormal basis $e_1,...,e_n$ of $\mathcal{V}$ a tensor $K$ has the form

$$\begin{aligned} K(e_1,e_1) = \lambda e_1, \quad K(e_1,e_j) = \mu e_j \\ K(e_j,e_j) = \mu e_1, \quad K(e_j,e_i) = 0 \end{aligned} \tag{15}$$

for $i \neq j$, $i,j > 1$, or, equivalently

$$K(X,Y) = (\lambda - 3\mu)g(X,e_1)g(Y,e_1)e_1 + \mu g(X,Y)e_1 + g(X,e_1)Y + \mu g(Y,e_1)X \tag{16}$$

for any vectors $X,Y \in \mathcal{V}$. In particular, the case where $\lambda = 3\mu$ appears on the Whitney sphere. For Lagrangian pseudospheres one has $\lambda = 2\mu$ (as in Example 3.3), for Lagrangian-umbilical submanifolds $\lambda = \mu$ (cf. [1]). Observe that $E = \mathrm{tr}\,_g K$ is equal to $(\lambda + (n-1)\mu)e_1$ and consequently $e_1 = E/\|E\|$. Since $K_{e_1}$ restricted to the orthogonal complement $\mathcal{D}$ to $e_1$ is a multiple of the identity, the orthonormal vectors $e_2,...,e_n$ can be chosen in $\mathcal{D}$ arbitrary.

If $X, Y$ are orthonormal vectors in $\mathcal{V}$ then

$$k(X \wedge Y) = \mu^2 + \mu(\lambda - 2\mu)(x_1^2 + y_1^2), \tag{17}$$

where $X = x_1 e_1 + X'$, $Y = y_1 e_1 + Y'$ for $X', Y' \in \mathcal{D}$. Observe that $x_1^2 + y_1^2 \le 1$. Indeed, we have $1 = x_1^2 + \varepsilon_1$, $1 = y_1^2 + \varepsilon_2$ and $x_1^2 y_1^2 = g(X', Y')^2 \le \varepsilon_1 \varepsilon_2$, where $\varepsilon_1 = \|X'\|^2$, $\varepsilon_2 = \|Y'\|^2$. The last inequality is equivalent to $(1-\varepsilon_1)(1-\varepsilon_2) \le \varepsilon_1\varepsilon_2$. Hence $\varepsilon_1 + \varepsilon_2 \ge 1$. We now have $2 = x_1^2 + y_1^2 + (\varepsilon_1 + \varepsilon_2) \ge x_1^2 + y_1^2 + 1$, which implies $x_1^2 + y_1^2 \le 1$.

One now sees that if $\mu(\lambda - 2\mu) \ge 0$ then $\mu^2 \le k(\pi) \le \mu(\lambda - \mu)$ for any vector plane $\pi$ in $\mathcal{V}$. Similarly, if $\mu(\lambda - 2\mu) \le 0$ then $\mu(\lambda - \mu) \le k(\pi) \le \mu^2$. In particular, if $\lambda = 3\mu$ then $\mu^2 \le k(\pi) \le 2\mu^2$. If $\lambda = 2\mu$ then $k(\pi) = \mu^2$ (as in Example 3.3), if $\lambda = \mu$ then $0 \le k(\pi) \le \mu^2$. If $\lambda = 0$ then $-\mu^2 \le k(\pi) \le \mu^2$.

Denote by $S^1$ the unit sphere $\{X \in \mathcal{V};\ \ g(X,X) = 1\}$ and by $\Phi$ the function

$$\Phi : S^1 \ni X \to C(X,X,X) = g(K(X,X),X) \in \mathbf{R}.$$

The function $\Phi$ attains its global maximum on $S^1$. This maximum is non-negative and equals 0 if and only if $K = 0$ on $\mathcal{V}$. But $\Phi$ may attain also local extrema on $S^1$. A local maximal value can be non-positive, see Example 3.12 below.

For orthonormal $U, W \in S^1$ we consider the mapping $\Phi(t) = \Phi(\cos t\, U + \sin t\, W)$. Then $\Phi(0) = \Phi(U)$,

$$\Phi'(0) = 3C(U,U,W),$$
$$\Phi''(0) = 3[2C(W,W,U) - C(U,U,U)]$$

and

$$\Phi'''(0) = 3[-7C(W,U,U) + 2C(W,W,W)].$$

Hence if $U \in S^1$ is a point where $\Phi$ attains its (maybe local) maximum and $W \in S^1$ is orthogonal to $U$ then

$$C(U,U,W) = 0, \qquad 2C(W,W,U) - C(U,U,U) \le 0 \tag{18}$$

and, if the equality holds in the last formula then $\Phi'''(0) = 0$ and consequently $C(W,W,W) = 0$.

The easiest situation which should be taken into account is when the sectional $K$-curvature is constant for all vector planes in $\mathcal{V}$. In this respect we have

**Lemma 3.5.** *Let $g$ be a scalar product on an $n$-dimensional vector space $\mathcal{V}$. Let $K$ be a symmetric $(1,2)$-tensor on $\mathcal{V}$ symmetric relative to $g$. If the sectional $K$-curvature is constant and equal to $A$ on $\mathcal{V}$ then there is an orthonormal basis $e_1, ..., e_n$ of $\mathcal{V}$ such that*

$$K(e_1, e_1) = \lambda_1 e_1, \quad K(e_1, e_i) = \mu_1 e_i \tag{19}$$

$$K(e_i, e_i) = \mu_1 e_1 + ... + \mu_{i-1} e_{i-1} + \lambda_i e_i, \tag{20}$$

*for $i = 2, ...n$ and*

$$K(e_i, e_j) = \mu_i e_j \tag{21}$$

*for some numbers $\lambda_i$, $\mu_i$ for $i = 1, ..., n-1$ and $j > i$. Moreover*

$$\mu_i = \frac{\lambda_i - \sqrt{\lambda_i^2 - 4A_{i-1}}}{2}, \tag{22}$$

$$A_i = A_{i-1} - \mu_i^2, \tag{23}$$

*for $i = 1, ..., n-1$ where $A_0 = A$.*

*If additionally $\mathrm{tr}_g K = 0$ then $A \le 0$, $\lambda_i$ and $\mu_i$ are expressed as follows*

$$\lambda_i = (n-i)\sqrt{\frac{-A_{i-1}}{n-i+1}}, \quad \mu_i = -\sqrt{\frac{-A_{i-1}}{n-i+1}}. \tag{24}$$

*In particular, in the last case the numbers $\lambda_i$, $\mu_i$ depend only on $A$ and the dimension of $\mathcal{V}$. Moreover, if $A < 0$ then $\lambda_i \neq 0$ and $\mu_i \neq 0$ for every $i$.*

Proof. Let $e_1 \in S^1$ be a point where $\Phi$ attains its maximum (maybe local). Then

$$g(K(e_1, e_1), U) = 0 \tag{25}$$

and

$$2g(K(e_1, U), U) - g(K(e_1, e_1), e_1) \le 0. \tag{26}$$

for each vector $U \in S^1$ orthogonal to $e_1$. The subspace $\{e_1\}^\perp$ is $K_{e_1}$-invariant, hence there is an orthonormal basis $e'_1, ..., e'_n$ of $\mathcal{V}$ diagonalizing $K_{e_1}$ such that $e'_1 = e_1$. Let $\lambda'_1 = \lambda_1$, $\lambda'_2$,..., $\lambda'_n$ be eigenvalues corresponding to the eigenvectors $e'_1, ..., e'_n$ of $K_{e_1}$. Taking in (13) $X = Z = e'_1$, $Y = W = e'_i$ for $2 \le i \le n$ we get

$$-A + \lambda_1 \lambda'_i - (\lambda'_i)^2 = 0. \tag{27}$$

If we regard this equality as an equation relative to $\lambda'_i$, we obtain at most two possible values $\lambda'_i = \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - 4A}}{2}$. By (26) we have $2\lambda'_i \le \lambda_1$. Therefore we may exclude the value $\frac{\lambda_1 + \sqrt{\lambda_1^2 - 4A}}{2}$. Set

$$\mu_1 = \frac{\lambda_1 - \sqrt{\lambda_1^2 - 4A}}{2}. \tag{28}$$

Note that under condition $\lambda_1 \ge 0$, if $A < 0$ then $\mu_1 < 0$, if $A = 0$ then $\mu_1 = 0$ and if $A > 0$ then $\mu_1 > 0$.

Let $\mathcal{D} = \{e_1\}^\perp$. Vectors belonging to this subspace will be denoted by $X', Y'$ etc. Denote by $K'$ the tensor on $\mathcal{D}$ defined as

$$K' = P \circ K_{|\mathcal{D} \times \mathcal{D}}, \tag{29}$$

where $P$ is the orthogonal projection onto $\mathcal{D}$. Note that this tensor has the same properties as $K$. First, it is symmetric and symmetric relative to $g$. Moreover

$$\begin{aligned} &A'(g(X', Z')g(Y', W') - g((X', W')g(Y', Z')) \\ &\quad = g(K(X', Z'), K(Y', W')) - g(K(X', W'), K(Y', Z')), \end{aligned}$$

for $X', Y', Z', W' \in \mathcal{D}$ and some number $A'$. Indeed, using (13) and the fact that $K(X', Z') = K'(X', Z') + \mu_1 g(X', Z') e_1$ we obtain

$$A(g(X', Z')g(Y', W') - g((X', W')g(Y', Z')) \tag{30}$$

$$= g(K(X', Z'), K(Y', W')) - g(K(X', W'), K(Y', Z')) \tag{31}$$

$$= \mu_1^2 (g(X', Z')g(Y', W') - g((X', W')g(Y', Z')) \tag{32}$$

$$+ g(K'(X', Z'), K'(Y', W')) - g(K'(X', W'), K'(Y', Z')). \tag{33}$$

Thus

$$A' = A - \mu_1^2. \tag{34}$$

In particular, if $A$ is negative then so is $A'$. Observe also that if the tensor $K$ is traceless then so is $K'$. Indeed, one has the following equalities

$$\begin{aligned}\sum_{i=2}^{n} K'(e'_i,e'_i) \quad &= \quad g(\sum_{i=2}^{n} K'(e'_i,e'_i),e'_2)e'_2+...+g(\sum_{i=2}^{n} K'(e'_i,e'_i),e'_n)e'_n\\ &= g(\sum_{i=2}^{n} K(e'_i,e'_i),e'_2)e'_2+...+g(\sum_{i=2}^{n} K(e'_i,e'_i),e'_n)e'_n\\ &= -g(K(e'_1,e'_1),e'_2)e'_2-...-g(K(e'_1,e'_1),e'_n)e'_n\\ &= -g(\lambda_1 e'_1,e'_2)e'_2-...-g(\lambda_1 e'_1,e'_n)e'_n=0.\end{aligned}$$

We can now apply the consideration from the beginning of the proof to the tensor $K'$ on $\mathcal{D}$. When replacing the basis $e'_2,...,e'_n$ by a new basis which is adapted to $K'$ as in the first part of the proof we use the fact that $K_{e_1|\mathcal{D}}$ is proportional to the identity. Using then the induction we get formulas (20)-(23).

Assume now that $K$ is traceless. Observe that in this case $A<0$ (if $K\neq 0$). Namely, take $Y=Z$ and the trace relative to $g$ in (13) at places of $X$ and $W$. We get the equality

$$A(1-n)g(Y,Y)=g(K_Y,K_Y), \tag{35}$$

for every $Y$ which shows that $A\le 0$ and $A=0$ if and only if $K=0$.

We have the following equalities characterizing $\lambda_1$ and $\mu_1$

$$\mu_1=\frac{\lambda_1-\sqrt{\lambda_1^2-4A}}{2},\qquad (n-1)\mu_1+\lambda_1=0.$$

Hence

$$\lambda_1=(n-1)\sqrt{\frac{-A}{n}},\qquad \mu_1=-\sqrt{\frac{-A}{n}}.$$

By induction we obtain formulas (24). $\square$

**Remark 3.6.** The expression for $K$ in the above proof is obtained in the following way. The vector $e_1$ is any vector at which $\Phi$ attains a local maximum on $S^1$, $e_2$ is any unit vector at which $\Phi_{|\mathcal{D}\cap S^1}$ attains its local maximum, etc. We construct a sequence $\lambda_1$, $\mu_1$, $A_1$, $\lambda_2$, $\mu_2$, $A_2$ etc. For a given $K$ the expression as in the above lemma is not unique in general. If $K$ is traceless, however, then the values $\lambda_i$ and $\mu_i$ are uniquely given. In particular, if $\Phi$ attains a local maximum on $S^1$ then its value $\lambda_1$ is equal to $(n-1)\sqrt{\frac{-A}{n}}$. Hence any local maximum of $\Phi$ on $S^1$ is its global maximum. The same deals with $\lambda_i$ for $i=2,...,n$.

Using the above proof one also gets

**Corollary 3.7.** *If in the above lemma $A=0$, that is, $[K,K]=0$, then there is an orthonormal basis $e_1,...,e_n$ of $\mathcal{V}$ such that*

$$K(e_i,e_i)=\lambda_i e_i,\quad K(e_i,e_j)=0 \tag{36}$$

*for $i,j=1,...n$ and $i\neq j$. If $[K,K]=0$ and $\mathrm{tr}\,_g K=0$ then $K=0$. If $K$ has expression (36 ) then $[K,K]=0$.*

In what follows we use some conventions established in the proof of Lemma 3.5. In particular, if some unit vector $e_1$ is fixed then the orthogonal complement to $e_1$ in $\mathcal{V}$ will be denoted by $\mathcal{D}$ and $K'$ will be defined by (29). Moreover by a maximum we shall mean a local maximum unless otherwise stated.

**Lemma 3.8.** *Let the sectional $K$-curvature on $\mathcal{V}$ be non-positive for every plane in $\mathcal{V}$ and $e_1 \in S^1$ be a point where $\Phi$ attains a maximum $\lambda_1 \neq 0$ on $S^1$. Then the sectional $K'$-curvature on $\mathcal{D}$ is also non-positive.*
*If moreover the sectional $K$-curvature is negative on $\mathcal{V}$ then the sectional $K'$-curvature on $\mathcal{D}$ is negative and strictly smaller than the $K$-sectional curvature on $\mathcal{D}$.*

Proof. We have an orthonormal basis $e_2, ..., e_n$ of $\mathcal{D}$ such that $e_1, e_2, ..., e_n$ is an orthonormal basis of eigenvectors of $K_{e_1}$. Let $\lambda_1, ..., \lambda_n$ be the corresponding eigenvalues of $K_{e_1}$. We have $2\lambda_j \leq \lambda_1$ for $j > 1$. Thus if $\lambda_1 < 0$ then $\lambda_j < 0$. If $\lambda_1 > 0$ we get $\lambda_j < \lambda_1$. By assumption we have

$$\begin{aligned} 0 \geq k(e_1 \wedge e_j) &= g(K(e_1, e_1), K(e_j, e_j)) - g(K_{e_1} e_j, K_{e_1} e_j) \\ &= g(\lambda_1 e_1, K(e_j, e_j)) - \lambda_j^2 = \lambda_j(\lambda_1 - \lambda_j). \end{aligned}$$

Therefore $\lambda_j \leq 0$ for every $j \geq 2$.

Assume that $\lambda_2, ..., \lambda_r$ are non-zero for some $r > 1$ and the next eigenvalues vanish. We can define a (positive definite) scalar product $G$ on the space $span\{e_2, ..., e_r\}$:

$$G(X', Y') = -(x_2 y_2 \lambda_2 + ... + x_r y_r \lambda_r), \tag{37}$$

where $X' = x_2 e_2 + ... + x_r e_r,\ Y' = y_2 e_2 + ... + y_r e_r$.

Let $X = x_2 e_2 + ... + x_n e_n,\ Y = y_2 e_2 + ... + y_n e_n$ be any two vectors of $\mathcal{D}$ and $X', Y'$ be their orthogonal projections onto the space $span\{e_2, ..., e_r)$. We have

$$\begin{aligned} K(X, Y) &= K'(X, Y) + g(K(X, Y), e_1) e_1 \\ &= K'(X, Y) + g(K_{e_1}(x_2 e_2 + ... + x_n e_n), y_2 e_2 + ... + y_n e_n) e_1 \\ &= K'(X, Y) + (x_2 y_2 \lambda_2 + ... + x_n y_n \lambda_n) e_1 \\ &= K'(X, Y) - G(X', Y') e_1. \end{aligned}$$

Thus

$$\begin{aligned} &g(K(X, X), K(Y, Y)) - g(K(X, Y), K(X, Y)) \\ &\quad = g(K'(X, X), K'(Y, Y)) - g(K'(X, Y), K'(X, Y)) \\ &\qquad\qquad + G(X', X') G(Y', Y') - G(X', Y')^2. \end{aligned}$$

Since $G(X', X')G(Y', Y') - G(X', Y')^2$ is non-negative by the Schwarz lemma, we have that

$$g(K'(X, X), K'(Y, Y)) - g(K'(X, Y), K'(X, Y)) \leq 0$$

if $g(K(X, X), K(Y, Y)) - g(K(X, Y), K(X, Y)) \leq 0$. The above consideration provides a proof of the lemma also in the case where all the eigenvalues $\lambda_2, ..., \lambda_n$ vanish.

If the sectional $K$-curvature is negative then $\lambda_j < 0$ for all $j \geq 2$. Thus $G$ is a scalar product on $\mathcal{D}$. Therefore, if $X, Y \in \mathcal{D}$ are orthonormal then $k'(X \wedge Y) < k(X \wedge Y)$, where $k'$ is the sectional $K'$-curvature. □

Analogously as above one gets

**Lemma 3.9.** *If the sectional $K$-curvature is non-negative or non-positive on $\mathcal{V}$ then the sectional $K'$-curvature $k'$ on $\mathcal{D}$ is not greater than the sectional $K$-curvature. More precisely, if $\pi$ is a plane in $\mathcal{D}$ then $k'(\pi) \leq k(\pi)$. If the sectional $K$-curvature is positive or negative on $\mathcal{V}$ then $k'(\pi) < k(\pi)$ for every plane in $\mathcal{D}$.*

From the proof of Lemma 3.22 we have the following useful observation

**Lemma 3.10.** *Let $e_1, ...., e_n$ be an orthonormal basis diagonalizing $K_{e_1}$ with corresponding eigenvalues $\lambda_1, ..., \lambda_n$. Then*

$$k(e_1 \wedge e_j) = \lambda_j(\lambda_1 - \lambda_j) \tag{38}$$

*for $j = 2, ..., n$. If $\lambda_1$ is a maximal value of $\Phi$ on $S^1$ and $\lambda_1 \geq 0$ then $\lambda_j \leq \lambda_1$. If $\lambda_1 > 0$ then $\lambda_j < \lambda_1$.*

**Proposition 3.11.** *Let $\lambda_1$ be a maximal value of $\Phi$ on $S^1$ attained at $e_1$ and $e_1, ..., e_n$ be an eigenbasis of $K_{e_1}$ with corresponding eigenvalues $\lambda_1, ..., \lambda_n$. If $\lambda_1 = 0$ then $k(e_1 \wedge e_j) \leq 0$ for every $j = 2, ..., n$. In particular, if the sectional $K$-curvature on $\mathcal{V}$ is positive for all planes then $\lambda_1 \neq 0$. If the structure $(g, K)$ is trace-free then $\lambda_1 \geq 0$ and $\lambda_1 = 0$ if and only if $K = 0$. For a trace-free structure the sectional $K$-curvature cannot be non-negative on $\mathcal{V}$.*

Proof. Assume that $\mathrm{tr}\,_g K = 0$. Then

$$\sum_{i=1}^{n} \lambda_i = 0. \tag{39}$$

If $\lambda_1 < 0$ then, because $2\lambda_j \leq \lambda_1$ for $j = 2, ..., n$, we have $\lambda_i < 0$ for all $i = 1, ..., n$. This contradicts (39). If $\lambda_1 = 0$ then $\lambda_j \leq 0$ for $j \geq 2$. By (39) all $\lambda_j = 0$. By the remark made after (18) we have that $K' = 0$ and consequently $K = 0$. Suppose that the sectional $K$-curvature is non-negative on $\mathcal{V}$ and $K \neq 0$. Then $\lambda_1 > 0$. By Lemma 3.10 we have $\lambda_1 - \lambda_j > 0$. By (39) there is $j > 1$ such that $\lambda_j < 0$. Hence, using (38), one gets the contradiction $k(e_1 \wedge e_j) < 0$. □

**Example 3.12.** It is possible that the sectional $K$-curvature is positive and $\lambda_1 < 0$. For instance, define $K$ on the standard Euclidean space $\mathbf{R}^2$ with the canonical basis $e_1, e_2$ by

$$K(e_1, e_1) = -3e_1, \quad K(e_1, e_2) = -2e_2, \quad K(e_2, e_2) = -2e_1.$$

One easily checks (using consideration before Lemma 3.5) that $\Phi$ attains a local maximum at $e_1$ and the $K$-curvature equals 2.

We shall need

**Lemma 3.13.** *Let $\Phi$ attain its maximum $\lambda_1$ at $e_1 \in S^1$ and $e_1, ..., e_n$ be an orthonormal eigenbasis of $K_{e_1}$ with corresponding eigenvalues $\lambda_i$, $i = 1, ..., n$. If $k(e_1 \wedge e_j) < \lambda_1^2/4$ for some $j = 2, ..., n$ then $2\lambda_j - \lambda_1 < 0$. In particular, if $\lambda_1 \neq 0$ and the sectional $K$-curvature is non-positive for all planes in $\mathcal{V}$ then $2\lambda_j - \lambda_1 < 0$ for every $j = 2, ..., n$.*

Proof. We know that $2\lambda_j - \lambda_1 \leq 0$ for every $j = 2, ..., n$. If $2\lambda_j - \lambda_1 = 0$ then $k(e_1 \wedge e_j) = \frac{\lambda_1^2}{4}$.

□

**Lemma 3.14.** *Let $\lambda_1$ be a maximal value of $\Phi$ on $S^1$ attained at $e_1 \in S^1$. Let $X \in S^1$ be orthogonal to $e_1$. Then $k(e_1 \wedge X) \leq \frac{\lambda_1^2}{4}$ and the equality holds if and only if $X$ is an eigenvector of $K_{e_1}$ with eigenvalue $\frac{\lambda_1}{2}$.*

Proof. Assume first that $X \in S^1$ is an eigenvector of $K_{e_1}$ with corresponding eigenvalue $\mu$. Then $k(e_1 \wedge X) = -\mu^2 + \mu\lambda_1$. Since the function $\mathbf{R} \ni t \to -t^2 + \lambda_1 t$ attains its maximum $\lambda_1^2/4$ for $t = \lambda_1/2$, we have that $k(e_1 \wedge X) \le \lambda_1^2/4$ and the equality holds if and only if $\mu = \lambda_1/2$.

As usual, let $e_1, ..., e_n$ be an orthonormal eigenbasis for $K_{e_1}$ and $\lambda_1, ..., \lambda_n$ be the corresponding eigenvalues. Let $X = x_2e_2 + ... + x_ne_n \in S^1$ be orthogonal to $e_1$ but not necessary an eigenvector of $K_{e_1}$. One now gets

$$\begin{aligned} k(e_1 \wedge X) &= g(K(e_1, e_1), K(X, X)) - g(K(e_1, X), K(e_1, X)) \\ &= \lambda_1 g(K_{e_1}(x_2e_2 + ... + x_ne_n), x_2e_2 + ... + x_ne_n) \\ &\quad -g(K_{e_1}(x_2e_2 + ... + x_ne_n), K_{e_1}(x_2e_2 + ... + x_ne_n)) \\ &= \lambda_1(x_2^2\lambda_2 + ... + x_n^2\lambda_n) - (x_2^2\lambda_2^2 + ... + x_n^2\lambda_n^2) \\ &= k(e_1 \wedge e_2)x_2^2 + ... + k(e_1 \wedge e_n)x_n^2 \\ &\le \tfrac{\lambda_1^2}{4}x_2^2 + ... + \tfrac{\lambda_1^2}{4}x_n^2 = \tfrac{\lambda_1^2}{4}. \end{aligned} \tag{40}$$

In this formula the equality holds if and only if for each $j = 2, ..., n$ either $x_j = 0$ or $\lambda_j = \lambda_1/2$. Assume that $x_2, ..., x_r$ are not zero and the next coordinates of $X$ vanish. Then $\lambda_2 = ... = \lambda_r = \lambda_1/2$ and one sees that $K_{e_1}X = \frac{\lambda_1}{2}X$. □

**Lemma 3.15.** *Let $\Phi$ attain its maximum at $e_1$ and $e_1, ..., e_n$ be an eigenbasis with corresponding eigenvalues $\lambda_1, ..., \lambda_n$. If $2\lambda_j - \lambda_1 < 0$ then for each $X \in S^1$ orthogonal to $e_1$ we have*

$$2C(X, X, e_1) - \lambda_1 < 0.$$

Proof. Let $X = x_2e_2 + ... + x_ne_n$. Then

$$\begin{aligned} 2C(X, X, e_1) &= 2g(K_{e_1}(x_2e_2 + ... + x_ne_n), x_2e_2 + ... + x_ne_n) \\ &= 2(\lambda_2x_2^2 + ... + \lambda_nx_n^2) < \lambda_1x_2^2 + ... + \lambda_1x_n^2 = \lambda_1. \end{aligned}$$

□

From the above lemmas we immediately get

**Proposition 3.16.** *Let $\lambda_1$ be a maximal value of $\Phi$ on $S^1$ attained at $e_1 \in S^1$ and $X \in S^1$ be orthogonal to $e_1$. Then $k(e_1 \wedge X) < \frac{\lambda_1^2}{4}$ if and only if $2C(X, X, e_1) - \lambda_1 <$ 0.*

By Lemma 3.14 we know that if the $K$-sectional curvature is constant then its value is less than or equal to $\frac{\lambda_1^2}{4}$ for any maximal value $\lambda_1$ on $S^1$.

**Proposition 3.17.** *Assume that the sectional $K$-curvature on $\mathcal{V}$ is constant and equal to $A = \frac{\lambda_1^2}{4}$ where $\lambda_1$ is a maximal value of $\Phi$ on $S^1$. Then there is an orthonormal basis $e_1, ..., e_n$ of $\mathcal{V}$ relative to which $K$ has expression as in* Example 3.3.

Proof. By Lemmas 3.5 and 3.14 we have $K_{e_1}X = \frac{\lambda_1}{2}X$ for any $X$ orthogonal to $e_1$. Since $\Phi$ attains a maximum $\lambda_1$ at $e_1$ and $2C(X, X, e_1) - \lambda_1 = 0$, by the observation made in the sentence containing (18) we know that $C(X, X, X) = 0$ for every $X$ orthogonal to $e_1$. □

Consider now the vector $E = \operatorname{tr}_g K$. If the sectional $K$-curvature is constant and equal to $\frac{\lambda_1^2}{4}$, where $\lambda_1$ is a maximum of $\Phi$ on $S^1$, then, by (10), $E = \frac{n+1}{2}\lambda_1e_1$. Therefore, if we have a statistical structure $(g, K)$ on a manifold $M$ of constant

sectional $K$-curvature equal to $\frac{\lambda_1^2}{4}$ at each point of $M$ then $\lambda_1$ is constant and $e_1$ is a smooth vector field on $M$.

Note that the assumption that $\lambda$ in Example 3.3 is a maximal value of $\Phi$ is not needed. We have the following characterizations of the structure from Example 3.3

**Theorem 3.18.** *Structures in* Example 3.3 *are characterized by the conjunction of the following conditions:*
*1) $E$ is an eigenvector of $K_E$,*
*2) $K_E$ restricted to the orthogonal complement to $E$ is a multiple of the identity,*
*3) the sectional $K$-curvature on $\mathcal{V}$ is a positive constant $A$,*
*4) $\|E\| = (n+1)\sqrt{A}$.*

Proof. Of course, if the structure is as in Example 3.3 then all conditions $1) - 4)$ are satisfied. Assume that the conditions $1) - 4)$ are fulfilled. By 1) and 2) we know that there exist numbers $\lambda$ and $\mu$ such that $K_{e_1}e_1 = \lambda e_1$ and $K_{e_1}e_i = \mu e_i$, for $i = 2, ..., n$, where $e_1 = \pm E/\|E\|$ and $e_1, ..., e_n$ is an orthonormal basis of $\mathcal{V}$. We choose the sign of $e_1$ in such a way that $\lambda \geq 0$. By 3), similarly as in the proof of Lemma 3.5, we obtain

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4B^2}}{2}, \tag{41}$$

where $A = B^2$, for some $B > 0$. In particular, we have $\lambda^2 - 4B^2 \geq 0$, which implies that $\lambda - 2B \geq 0$. By 4) we have

$$(n-1)\mu + \lambda = (n+1)B. \tag{42}$$

Inserting (41) into (42) we get

$$\pm(n-1)\sqrt{\lambda - 2B}\sqrt{\lambda + 2B} = -(n+1)\sqrt{\lambda - 2B}\sqrt{\lambda - 2B}.$$

Assume that $\lambda \neq 2B$. Then $\mp(n-1)\sqrt{\lambda + 2B} = (n+1)\sqrt{\lambda - 2B}$ and consequently $(n-1)^2(\lambda + 2B) = (n+1)^2(\lambda - 2B)$. It follows that

$$\lambda = \frac{n^2+1}{n}B.$$

Inserting this into (41) one gets $\mu = nB$ or $\mu = B/n$. Using now (42) we obtain contradictions. Therefore $\lambda = 2B$ and, by (41) $\mu = \lambda/2$. It follows that $A = \lambda^4/4$. We can now go back to the proof of Lemma 3.5. By (34) we see that the sectional $K'$-curvature on $\mathcal{D}$ vanishes. Hence $K'$ has expression as in Corollary (3.7). But $E$ is proportional to $e_1$, hence $K' = 0$ and consequently $K$ has expression as in Example 3.3. □

**Theorem 3.19.** *Let $(g, K)$ be a statistical structure on $M$ such that at each point $p$ of $M$ the tensor $K_p$ is as in Theorem* 3.18*. If $\hat{\nabla}K$ is symmetric and* $\operatorname{div} E$ *is constant then the sectional curvature (for $g$) by any plane containing $E$ is non-positive. If $\hat{\nabla}E = 0$ then $\hat{\nabla}K = 0$ on $M$.*

Proof. We can assume that $M$ is connected. Since $\hat{\nabla}K$ is symmetric, the sectional $K$-curvature is constant on $M$. We have $E = \Lambda e_1$ where $\Lambda$ is a smooth function and $e_1$ is a smooth unit vector field on $M$. $\lambda$ is a constant function on $M$. Locally we can extend $e_1$ to a smooth orthonormal frame $e_1, ..., e_n$. In such a frame $K$ has expression as in Example 3.3. Then $\Lambda = (n+1)\lambda/2$. Let $\hat{\nabla}_{e_i}e_j = \sum_{k=1}^n \omega_j^k(e_i)e_k$.

By a straightforward computation one gets for mutually different $i, j, l \geq 2$

$$
\begin{aligned}
&(\hat{\nabla}_{e_1} K)(e_1, e_1) = 0 \\
&(\hat{\nabla}_{e_i} K)(e_i, e_i) = \tfrac{\lambda}{2} \textstyle\sum_{k \neq 1} \omega_1^k(e_i) e_k + \lambda \omega_1^i(e_i) e_i \\
&(\hat{\nabla}_{e_i} K)(e_1, e_1) = 0 \\
&(\hat{\nabla}_{e_1} K)(e_i, e_1) = 0 \\
&(\hat{\nabla}_{e_1} K)(e_i, e_i) = \tfrac{\lambda}{2} \textstyle\sum_{k \neq 1, i} \omega_1^k(e_1) e_k + \tfrac{3}{2} \lambda \omega_1^i(e_1) e_i \\
&(\hat{\nabla}_{e_i} K)(e_1, e_i) = 0 \\
&(\hat{\nabla}_{e_1} K)(e_i, e_j) = \tfrac{\lambda}{2} \omega_1^i(e_1) e_j + \tfrac{\lambda}{2} \omega_1^j(e_1) e_i \\
&(\hat{\nabla}_{e_i} K)(e_1, e_j) = 0 \\
&(\hat{\nabla}_{e_i} K)(e_j, e_j) = \tfrac{\lambda}{2} \textstyle\sum_{k \neq 1} \omega_1^k(e_i) e_k + \lambda \omega_1^j(e_i) e_j \\
&(\hat{\nabla}_{e_j} K)(e_i, e_j) = \tfrac{\lambda}{2} \omega_1^j(e_j) e_i + \tfrac{\lambda}{2} \omega_1^i(e_j) e_j \\
&(\hat{\nabla}_{e_i} K)(e_j, e_l) = \tfrac{\lambda}{2} \omega_1^j(e_i) e_l + \tfrac{\lambda}{2} \omega_1^l(e_i) e_j \\
&(\hat{\nabla}_{e_j} K)(e_i, e_l) = \tfrac{\lambda}{2} \omega_1^i(e_j) e_l + \tfrac{\lambda}{2} \omega_1^l(e_j) e_i .
\end{aligned}
\tag{43}
$$

One now sees that if $\hat{\nabla} K$ is symmetric then $\hat{\nabla}_{e_1} e_1 = 0$, and $\hat{\nabla}_{e_i} e_1 = \alpha e_i$ for some function $\alpha$ for every $i = 2, ..., n$. It implies that

$$g(\hat{R}(e_i, e_1)e_1, e_i) = -(e_1 \alpha + 2\alpha^2)$$

for every $i = 2, ..., n$. Since $\operatorname{div} E = \frac{n^2-1}{2} \lambda \alpha$ is constant, the function $\alpha$ is constant if $\operatorname{div} E$ is constant. Consequently $g(\hat{R}(e_i, e_1)e_1, e_i) = -2\alpha^2$. If $\hat{\nabla} E = 0$ then $\hat{\nabla} e_1 = 0$ and formulas (43) imply $\hat{\nabla} K = 0$. □

If $J$ is an endomorphism of $\mathcal{V}$ and $T$ is a tensor on $\mathcal{V}$ then $J \cdot T$ will mean that $J$ acts as a differentiation on $T$. If $\mathcal{R}$ is a tensor of type $(1,3)$ and $\mathcal{R}(X, Y)$ denotes the endomorphism determined by $\mathcal{R}$ then the equality $\mathcal{R} \cdot T = 0$ means that $\mathcal{R}(X, Y) \cdot T = 0$ for every $X, Y \in \mathcal{V}$. If $X \in \mathcal{V}$ then $\mathcal{R}X = 0$ means that $\mathcal{R}(Y, Z)X = 0$ for every $Y, Z \in \mathcal{V}$. The same convention will be used for tensor fields on manifolds.

**Lemma 3.20.** *Let $J$ be an endomorphism of $\mathcal{V}$ such that $J \cdot g = 0$, where $J$ is regarded as a differentiation. If the sectional $K$-curvature is negative for every plane of $\mathcal{V}$ and $J \cdot K = 0$ then $J = 0$.*

Proof. As usual take $e_1 \in S^1$ where $\Phi$ attains its maximum and an orthonormal eigenbasis $e_1, ..., e_n$ of $K_{e_1}$ with corresponding eigenvalues $\lambda_1, ..., \lambda_n$. By Lemma 3.13 we know that $2\lambda_i - \lambda_1 < 0$ for all $i = 2, .., n$. Using the fact that $J$ is skew-symmetric relative to $g$ we obtain

$$
\begin{aligned}
0 = (J \cdot K)(e_1, e_1) &= J(K(e_1, e_1)) - 2K(Je_1, e_1) \\
&= \lambda_1 \sum_{j=2}^{n} g(Je_1, e_j) e_j - 2K\Big(\sum_{j=2}^{n} g(Je_1, e_j) e_j, e_1\Big) \\
&= \sum_{j=2}^{n} (\lambda_1 - 2\lambda_j) g(Je_1, e_j) e_j .
\end{aligned}
$$

Using also the fact that $g(Je_1, e_1) = 0$, we get $Je_1 = 0$. In particular, the orthogonal complement $\mathcal{D}$ to $e_1$ in $\mathcal{V}$ is $J$-invariant. Let $K'$ be given by (29) and $J'$ stands for

the restriction of $J$ to $\mathcal{D}$. For $X', Y' \in \mathcal{D}$ we get (using the skew-symmetry of $J$, the condition $J \cdot K = 0$ and the equality $Je_1 = 0$)

$$\begin{aligned}(J' \cdot K')(X', Y') &= J(K(X', Y') - g(K(X', Y'), e_1)e_1) \\ &\quad -K(JX', Y') + g(K(JX', Y'), e_1)e_1 - K(X', JY') + g(K(X', JY'), e_1)e_1 \\ &= (J \cdot K)(X', Y') + g(J(K(X', Y')), e_1)e_1 = 0.\end{aligned}$$

By Lemma 3.9 we see that $K' \neq 0$ and the sectional $K'$-curvature is negative on $\mathcal{D}$. We can now apply the same as above arguments for the objects $K', J'$ on $\mathcal{D}$ and continue the proof using induction. □

Using Lemma 3.13 and the first part of the proof of Lemma 3.20 we obtain

**Lemma 3.21.** *Let $J$ be an endomorphism of $\mathcal{V}$ such that $J \cdot g = 0$, where $J$ is treated as a differentiation. Assume that $\lambda_1 \neq 0$ is a maximal value of $\Phi$ on $S^1$ attained at $e_1 \in S^1$. If the sectional $K$-curvature is smaller than $\lambda_1^2/4$ for every plane of $\mathcal{V}$ and $J \cdot K = 0$ then $Je_1 = 0$.*

**Theorem 3.22.** *If the sectional $K$-curvature is non-positive on $\mathcal{V}$ and $[K, K] \cdot K = 0$ then the sectional $K$-curvature vanishes on $\mathcal{V}$.*

Proof. We can modify the proof of Lemma 3.20. Assume that $K \neq 0$. Let $\lambda_1 > 0$ be a maximal value of $\Phi$ on $S^1$ attained at $e_1$ and $e_1, ..., e_n$ be an eigenbasis of $K_{e_1}$ with corresponding eigenvalues $\lambda_1, ..., \lambda_n$. By Lemma 3.13 we have $2\lambda_j - \lambda_1 \neq 0$. As in the proof of Lemma 3.20 we get $[K, K]e_1 = 0$. It follows that $\lambda_j(\lambda_1 - \lambda_j) = 0$ for every $j \geq 2$, hence $\lambda_j = 0$ for every $j \geq 2$ (because $\lambda_j < \lambda_1$ if $\lambda_1 > 0$, see Lemma 3.10). Thus $K(e_1, X') = 0$ and consequently $g(e_1, K(X', Y')) = 0$ for every $X', Y' \in \mathcal{D}$. Let $X, Y$ be any vectors of $\mathcal{V}$ and $X = x_1e_1 + X', Y = y_1e_1 + Y'$ for some $X', Y' \in \mathcal{D}$. We now have

$$\begin{aligned}&g(K(X, X), K(Y, Y)) - g(K(X, Y), K(X, Y)) \\ &= g(x_1^2\lambda_1e_1 + K(X', X'), y_1^2\lambda_1e_1 + K(Y', Y')) \\ &\quad -g(x_1y_1\lambda_1e_1 + K(X', Y'), x_1y_1\lambda_1e_1 + K(X', Y')) \\ &= g(K'(X', X') + g(K(X', X'), e_1)e_1, K'(Y', Y') + g(K(Y', Y')e_1)e_1) \\ &\quad -g(K'(X', Y') + g(K(X', Y')e_1)e_1, K'(X', Y') + g(K(X', Y')e_1)e_1) \\ &= g(K'(X', X'), K'(Y', Y')) - g(K'(X', Y'), K'(X', Y')),\end{aligned}$$

that is,

$$\begin{aligned}&g(K(X, X), K(Y, Y)) - g(K(X, Y), K(X, Y)) \\ &\quad = g(K'(X', X'), K'(Y', Y')) - g(K'(X', Y'), K'(X', Y')).\end{aligned} \tag{44}$$

It follows that the sectional $K$-curvature vanishes on $\mathcal{V}$ if $K' = 0$ and if $K' \neq 0$ the sectional $K'$-curvature on $\mathcal{D}$ is non-positive. In the last case we argue as above for the structure $K'$ on $\mathcal{D}$. We get $[K', K']e_2 = 0$ and $K'(e_2, X'') = 0$ for every $X'' \in \mathcal{D}$ orthogonal to $e_2$, where $e_2 \in S^1 \cap \mathcal{D}$ is a point where $\Phi_{|S^1 \cap \mathcal{D}}$ attains its positive maximal value. Then we continue the proof by induction using the same type of arguments as above and we obtain the expression for $K$ as in Corollary 3.7. □

As consequences of Theorem 3.22 we obtain

**Corollary 3.23.** *If $(g, K)$ is a Hessian structure on $M$ with non-negative sectional curvature of $g$ and such that $\hat{R} \cdot K = 0$ then $g$ is flat.*

Lemma 3.20 yields

**Theorem 3.24.** *If $(g, K)$ is a statistical structure on a manifold $M$, the sectional $K$-curvature is negative on $M$ and $\hat{R} \cdot K = 0$ then $g$ is flat.*

In the following theorem $M$ is $n$-dimensional and the complex space form has complex dimension $n$.

**Theorem 3.25.** *If $M$ is a totally real submanifold of the complex space form of holomorphic sectional curvature $4c$, the sectional curvature of $M$ is smaller than $c$ on $M$ and $\hat{R} \cdot K = 0$, where $K$ is the second fundamental tensor of the submanifold then $\hat{R} = 0$.*

Proof. We have the following Gauss equation for a totally real submanifold in the complex space form

$$c[g(Y,Z)X - g(X,Z)Y] = \hat{R}(X,Y)Z - [K_X, K_Y]Z \tag{45}$$

for every $X, Y, Z$ tangent to $M$. Hence the sectional $K$-curvature equals to the difference between the sectional curvature for $g$ and $c$. Therefore the assumption of the theorem says that the sectional $K$-curvature is negative so we can use Lemma 3.20. □

**Theorem 3.26.** *Let $(g, K)$ be a statistical structure on a connected manifold $M$ and $g$ has constant sectional curvature. If at some point $p$ of $M$ the equality $\hat{R} \cdot K = 0$ holds and the sectional $K$-curvature is positive on $T_pM$ and strictly smaller than the maximal value of $\Phi$ on the unit sphere in $T_pM$ then $g$ is a flat metric.*

Proof. By the proof of Lemma 3.20 we have that $\hat{R}_p e_1 = 0$. Hence $\hat{R} = 0$. □

## 4. A smoothness lemma and its consequences for statistical structures of constant sectional $K$-curvature

Although for a statistical structure $(g, K)$ with constant $K$-sectional curvature on a manifold $M$ at each point of $M$ we can find an orthonormal frame for which $K$ has expression as in Lemma 3.5, it is not possible, in general, to find a smooth orthonormal local frame relative to which $K$ has this nice expression. Even to find a smooth local vector field $e_1$ at which $\Phi$ attains a maximum makes a problem. We shall now prove (see Lemma 4.2 below) that in some cases it is possible. Since in the proof we shall use the multiple Lagrange method, we can only get a vector field at which $\Phi$ attains a local maximum (even if we start with a global maximum at some point $p \in M$). It is why we have used local maxima in our considerations, for instance in Lemma 3.5. Since the author of this paper was unable to find references for Lemma 4.2 with a rigorous proof, we provide a detailed proof. We shall use Lemma 4.2 only for the cubic form of statistical structures, but we formulate and prove the result for symmetric forms of any degree.

We shall start with a topological lemma

**Lemma 4.1.** *Let $\pi : \mathcal{H} \to M$ be a locally trivial bundle with a compact standard fiber $H$ and let $\psi : \mathcal{H} \to T$ be a continues mapping into a topological space $T$. If $\mathcal{H}_p \subset \psi^{-1}(B)$ for some open subset $B \subset T$ then there is a neighborhood $\mathcal{U}$ of $p$ in $M$ such that $\bigcup_{x \in \mathcal{U}} \mathcal{H}_x \subset \psi^{-1}(B)$.*

Proof. We can assume that in some neighborhood $M'$ of $p$ the bundle of the shape $M' \times H$. For every $v \in \mathcal{H}_p = \{p\} \times H$ there is a neighborhood $\mathcal{U}_v$ of $v$ in $\mathcal{H}$ such that $\psi(\mathcal{U}_v) \subset B$. We can assume that $\mathcal{U}_v = U_v \times D_v$, where $D_v$ is an open subset in $H$ and $U_v$ is an open neighborhood of $p$. Of course $\bigcup_{v\in\mathcal{H}_p} D_v$ contains $H$. We choose a finite subcovering $D_{v_1}, ..., D_{v_r}$ of $H$ and take $\mathcal{U} = \bigcap_{i=1}^r U_{v_i}$. Let $(x, v) \in \mathcal{U} \times H$. Then $(x, v) \in \mathcal{U}_{v_i}$ for some $i = 1, ..., r$. Hence $\psi(U) \subset B$. $\square$

By a Riemannian vector bundle we mean a vector bundle $\mathcal{W} \to M$ for which each fibre $\mathcal{W}_p$ has a scalar product $g_p$ and the assignment $p \to g_p$ is smooth.

**Lemma 4.2.** *Let $\mathcal{W}$ be a Riemannian vector bundle over $M$ and $U\mathcal{W}$ be its unit sphere bundle. Assume that $C$ is a smooth field of symmetric $(0, k)$-tensors on $\mathcal{W}$ and $\Phi$ is defined as follows $\Phi : U\mathcal{W} \ni X \to C(X, ..., X) \in \mathbf{R}$. Assume that at each point $p \in M$ the function $\Phi_p = \Phi_{|U\mathcal{W}_p}$ has the following property:*
*(*) If $\Phi_p$ attains its local maximum on $U\mathcal{W}_p$ at $X_0$ then $(k-1)C(U, U, X_0, ..., X_0) - C(X_0, ..., X_0) \neq 0$ for every $U \in U\mathcal{W}_p$ orthogonal to $X_0$.*
*Then for every $p \in M$ there is a smooth unit section $e_1$ of $\mathcal{W}$, defined in some neighborhood of $p$, such that $\Phi_x$ attains its (local) maximum on $U\mathcal{W}_x$ at $e_1(x)$ for each $x$ from this neighborhood.*

Proof. Let $p \in M$ be a fixed point. Denote by $n$ the rank of the bundle $\mathcal{W}$. Let $e_1$ be a point of $U\mathcal{W}_p$ at which $\Phi$ attains a local maximum. Then

$$C(U, e_1, ..., e_1) = 0 \tag{46}$$

and

$$(k-1)C(U, U, e_1, ..., e_1) - C(e_1, ..., e_1) < 0 \tag{47}$$

for any $U \in U\mathcal{W}_p$ orthogonal to $e_1$. Let $G$ be a symmetric 2-form on $\mathcal{W}_p$ given by $G(X, Y) = C(X, Y, e_1, ..., e_1)$. Since $G(U, e_1) = 0$ for every $U$ orthogonal to $e_1$ there is an orthogonal basis $e_1, ..., e_n$ of $\mathcal{W}_p$ diagonalizing $G$. Let $\lambda_1$,..., $\lambda_n$ be eigenvalues corresponding to the basis $e_1, ..., e_n$. We have $(k-1)\lambda_i - \lambda_1 < 0$ for $i = 2, ..., n$. Extend the orthonormal frame to any local orthonormal frame, say $E_1, ..., E_n$ in a neighborhood $\mathcal{U}$ of $p$. Let $C_{i_1...i_k}$ be the coordinates of the form $C$ relative to this local frame, that is, $C_{i_1...i_k} = C(E_{i_1}, ..., E_{i_k})$. Consider functions $f : \mathcal{U} \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ defined as follows

$$f(x, y_1, ..., y_n, \lambda) = \sum_{i_1,...,i_k=1}^{n} C_{i_1...i_k}(x) y_{i_1} \cdots y_{i_k} - \lambda(y_1^2 + ... + y_n^2 - 1).$$

By the Lagrange method one knows that at a fixed point $x \in \mathcal{U}$, the extrema of the function $\sum_{i_1,...,i_k=1}^{n} C_{i_1...i_k}(x) y_{i_1} \cdots y_{i_k}$ on the sphere $y_1^2 + ... + y_n^2 - 1 = 0$ are in the set described by the system of equations

$$\begin{aligned}
\frac{\partial f}{\partial y_1} &= k\sum_{i_2,\dots,i_k=1}^{n} C_{1i_2\dots i_k}(x)y_{i_2}\cdots y_{i_k} - 2\lambda y_1 = 0\\
\frac{\partial f}{\partial y_2} &= k\sum_{i_2,\dots,i_k=1}^{n} C_{2i_2\dots i_k}(x)y_{i_2}\cdots y_{i_k} - 2\lambda y_2 = 0\\
&\ \ .\\
&\ \ .\\
&\ \ .\\
\frac{\partial f}{\partial y_n} &= k\sum_{i_2,\dots,i_k=1}^{n} C_{ni_2\dots i_k}(x)y_{i_2}\cdots y_{i_k} - 2\lambda y_n = 0\\
\frac{\partial f}{\partial \lambda} &= -(y_1^2+\dots+y_n^2-1) = 0
\end{aligned}$$

Define the functions

$$F_i(x,y_1,\dots y_n,\lambda) = k\sum_{i_2,\dots,i_k=1}^{n} C_{ii_2\dots i_k}(x)y_{i_2}\cdots y_{i_k} - 2\lambda y_i \tag{48}$$

for $i=1,\dots,n$ and

$$F_{n+1}(x,y_1,\dots,y_n,\lambda) = y_1^2+\dots+y_n^2-1.$$

Set $y_{n+1}=\lambda$. Let $F=(F_1,\dots,F_{n+1}) : \mathcal{U}\times\mathbf{R}^{n+1}\to\mathbf{R}^{n+1}$.

We want to find smooth functions $y_1(x),\dots,y_n(x),\lambda(x)$, which satisfy the equation $F(x,y_1(x),\dots,y_{n+1}(x),\lambda(x))=(0,\dots,0)$ and satisfy the initial conditions $y_1(p)=1, y_2(p)=0\dots,y_n(p)=0$, $y_{n+1}(p)=\lambda(p)=\frac{k}{2}C_{1\dots1}=\frac{k}{2}\lambda_1$. The initial conditions follow from the fact that the vector $e_1=(1,0,\dots,0)$ is among solutions of the above system of equations and $\lambda(p)$ can be computed from the first equation of the system. We shall now use the implicit function theorem. To this aim, it is sufficient to check that

$$\det\left(\frac{\partial F_i}{\partial y_j}\right)(p,e_1,\frac{k}{2}\lambda_1)\neq 0.$$

We have

$$\frac{\partial F_i}{\partial y_j} = k(k-1)\sum_{i_3,\dots,i_k=1}^{n} C_{iji_3\dots i_k}(x)y_{i_3}\cdots y_{i_k} - 2\delta_{ij}\lambda$$

for $i,j=1,\dots,n$. It follows that at the initial values we have

$$\frac{\partial F_i}{\partial y_j}(p,(1,0,\dots,0),\frac{k}{2}\lambda_1) = k[(k-1)C_{ij1\dots1}(p)-\delta_{ij}\lambda_1]$$

for $i,j=1,\dots,n$. In particular

$$\frac{\partial F_1}{\partial y_1}(p,(1,0,\dots,0),\frac{k}{2}\lambda_1) = k[(k-1)C_{1\dots1}(p)-\lambda_1] = k(k-2)\lambda_1,$$

$$\frac{\partial F_i}{\partial y_i}(p,(1,0,\dots,0),\frac{k}{2}\lambda_1) = k[(k-1)C_{ii1\dots1}(p)-\lambda_1)] = k[(k-1)\lambda_i-\lambda_1],$$

$$\frac{\partial F_i}{\partial y_j}(p,(1,0,\dots,0),\frac{k}{2}\lambda_1) = 0$$

for $i \neq j$, $i, j = 2, ..., n$. Moreover
$$\frac{\partial F_1}{\partial y_{n+1}}(p, (1, 0, ..., 0), \frac{k}{2}\lambda_1) = -2,$$
$$\frac{\partial F_j}{\partial y_{n+1}}(p, (1, 0, ..., 0), \frac{k}{2}\lambda_1) = 0$$
for $j = 2, ..., n$;
$$\frac{\partial F_{n+1}}{\partial y_1}(p, (1, 0, ..., 0), \frac{k}{2}\lambda_1) = 2,$$
$$\frac{\partial F_{n+1}}{\partial y_i}(p, (1, 0, ..., 0), \frac{k}{2}\lambda_1) = 0$$
for $i = 2, ..., n$;
$$\frac{\partial F_{n+1}}{\partial y_{n+1}}(p, (1, 0, ..., 0), \frac{k}{2}\lambda_1) = 0.$$
It is now clear that $\det \left(\frac{\partial F_i}{\partial y_j}\right)(p, e_1, \frac{k}{2}\lambda_1) \neq 0$.

Let $y_1(x), ..., y_n(x), \lambda(x)$ be the solution of of our implicit function problem. Denote by $e_1$ the section of $\mathcal{W}$ given by $y_1E_1 + ... + y_nE_n$. Since the condition $F(x, e_1(x), \lambda(x)) = 0$ is satisfied, at each point of some neighborhood $\mathcal{U}'$ of $p$, we have that $C(U, e_1, ..., e_1) = 0$ for every $U$ orthogonal to $e_1$, $U \in U\mathcal{W}_x$ at each $x \in \mathcal{U}'$. To see this it is sufficient to multiply each $F_i(x, e_1(x), \lambda(x))$ by $U_i$ (where $U = U_1E_1 + ... + U_nE_n$) and make summation relative to $i = 1, ..., n$.

Using now Lemma 4.1 one sees that since $C(e_1, ..., e_1) > 2C(U, U, e_1, ..., e_1)$ for each $U \in U\mathcal{W}_p$, there is a neighborhood $\mathcal{U}'' \subset \mathcal{U}'$ of $p$ such that $C(e_1, ..., e_1) > 2C(U, U, e_1, ..., e_1)$ for each $U \in U\mathcal{W}_x$ and $x \in \mathcal{U}''$. Indeed, it is sufficient to take as $\mathcal{H}$ the bundle $U\mathcal{W}_{|\mathcal{U}'} \cap \mathcal{D}$, where $\mathcal{D}$ is the orthogonal complement to $e_1$ in the bundle $\mathcal{W}_{|\mathcal{U}'}$ and define $\psi$ as the mapping
$$\psi : \mathcal{H} \ni V \to C(V, V, e_1, ..., e_1) \in \mathbf{R}.$$

It follows that for every $x \in \mathcal{U}''$ the mapping $\Phi_x$ attains at $e_1(x)$ a local maximum. □

**Lemma 4.3.** *Let $(g, K)$ be a statistical structure on a manifold $M$ and its sectional $K$-curvature is constant. Assume that for each point $p$ of $M$ there is a local orthonormal frame $e_1, ..., e_n$ around $p$ relative to which $K$ has expression as in* Lemma 3.5*; $\lambda_i$, $\mu_i$ are constant and $\lambda_i - 2\mu_i \neq 0$ for every $i = 1, ..., n-1$. If $\hat\nabla K$ is symmetric then $\hat\nabla e_j = 0$ for every $j = 1, ..., n$. In particular, $\hat R = 0$ and $\hat\nabla K = 0$ on $M$.*

Proof. For every $j > 1$ we have
$$g((\hat\nabla_{e_j}K)(e_1, e_1), e_1) = 0, \quad g((\hat\nabla_{e_1}K)(e_j, e_1), e_1) = (\lambda_1 - 2\mu_1)\omega_1^j(e_1)$$
for every $j > 1$. By the symmetry of $\hat\nabla K$ one now has $\hat\nabla_{e_1}e_1 = 0$. Assume now that $k > 1$ and $j > 1$. Using the fact that $\hat\nabla_{e_1}e_1 = 0$ we obtain
$$g((\hat\nabla_{e_j}K)(e_k, e_1), e_1) = (\lambda_1 - 2\mu_1)\omega_1^k(e_j), \qquad g((\hat\nabla_{e_1}K)(e_k, e_j), e_1) = 0.$$
Hence $\hat\nabla e_1 = 0$. Assume now that $\hat\nabla e_1 = 0, ..., \hat\nabla e_{i-1} = 0$. In particular, $\omega_i^k(e_i) = 0$ for every $k < i$. By a straightforward computation one gets
$$g((\hat\nabla_{e_j}K)(e_i, e_i), e_i) = 0, \quad g((\hat\nabla_{e_i}K)(e_j, e_i), e_i) = (\lambda_i - 2\mu_i)\omega_i^j(e_i)$$

for $j > i$. Hence $\hat\nabla_{e_i} e_i = 0$. For $k > i$ and any $j$ we obtain $g((\hat\nabla_{e_j} K)(e_k, e_i), e_i) = (\lambda_i - 2\mu_i)\omega_i^k(e_j)$. In both cases: $j > i$ and $j < i$ one gets $g((\hat\nabla_{e_i} K)(e_k, e_j), e_i) = 0$. Thus $\hat\nabla e_i = 0$ for $i = 1, ..., n-1$. It is now clear that $\hat\nabla e_n = 0$ as well. □

**Theorem 4.4.** *Let $(g, K)$ be a trace-free statistical structure on a manifold $M$ with symmetric $\hat\nabla K$. If the sectional $K$-curvature is constant then either $K = 0$ or $\hat R = 0$ and $\hat\nabla K = 0$.*

Proof. Assume that $K \neq 0$. It means that $K_x \neq 0$ at every point $x$ of $M$, because the sectional curvature is constant and $K$ is traceless. At each point of $M$ the tensor $K$ has the expression as in Lemma 3.5 with values $\lambda_i$, $\mu_i$ given by (24) (non-zero and constant on $M$). Moreover, $\lambda_i - 2\mu_i \neq 0$. By Lemma 4.2 we know that for each $p \in M$ there is a unit vector field $e_1$ in a neighborhood of $p$ such that $\Phi_x$ attains a maximum $\lambda_1$ at $(e_1)_x$ for each point $x$ of this neighborhood. We take the orthogonal complement $\mathcal{D}$ to $e_1$ in the domain of $e_1$. By Lemma 4.2 one gets a smooth vector field $e_2$ at which $\Phi_{|\mathcal{D}}$ attains a maximum (at each point of a domain of $e_2$) and then we proceed inductively. In this way we obtain a smooth frame field $e_1, ...., e_n$ relative to which $K$ has expression as in Lemma 3.5 with constant $\lambda_i$, $\mu_i$ for $i = 1, ..., n$. Using now Lemma 4.3 completes the proof. □

**Remark 4.5.** Particular versions of the above theorem have been given for minimal Lagrangian space forms in complex space forms, see [2] and for affine hyperspheres with constant sectional curvature, see Theorem 2.2.3.18 in [4].

We shall say that a tensor $K$ of type $(1, 2)$ is non-degenerate if the mapping $X \to K_X$ is a monomorphism.

**Theorem 4.6.** *Assume that $[K, K] = 0$ on a statistical manifold $(M, g, K)$, $\hat\nabla K$ is symmetric and $\hat\nabla E = 0$. If $K$ is non-degenerate at each point of $M$ then $\hat R = 0$ and $\hat\nabla K = 0$ on $M$.*

Proof. At each point $p \in M$ the tensor $K_p$ can be expressed as in Corollary 3.7 and all $\lambda_i$ are non-zero. Let $p$ be a fixed point of $M$. By Lemma 4.2 there is a local unit vector field $e_1$ around $p$ such that $\Phi$ attains its local maximum on $U_xM$ for every $x$ from a neighborhood of $p$. Let $\lambda_1 = C(e_1, e_1, e_1)$. Take the distribution $\mathcal{D}$ orthogonal to $e_1$. We now take $e_2$ where $\Phi$ restricted to $\mathcal{D}_p$ attains its maximum $\lambda_2$. Again we can apply Lemma 4.2 and get a unit smooth local vector field $e_2$ in a neighborhood of $p$ such that $\Phi_{|\mathcal{D}\cap U_xM}$ attains a maximum at $e_2$ for each $x$ from this neighborhood. Continuing this process and using the proof of Lemma 3.5 we obtain a smooth orthonormal local frame $e_1, ..., e_n$ in a neighborhood of $p$ such that $K(e_i, e_j) = \delta_{ij}\lambda_i e_i$ for $i, j = 1, ..., n$. The functions $\lambda_i = C(e_i, e_i, e_i)$ are smooth.

We shall now use the assumption that $\hat\nabla K$ is symmetric in order to show that $\hat\nabla e_j = 0$ for all $j = 1,,,, n$. For $i \neq j$ we have

$$(\hat\nabla_{e_i} K)(e_j, e_j) = (e_i\lambda_j)e_j + \lambda_j \sum_{l\neq j} \omega_j^l(e_i)e_l,$$

$$(\hat\nabla_{e_j} K)(e_i, e_j) = -\omega_i^j(e_j)\lambda_j e_j - \omega_j^i(e_j)\lambda_i e_i.$$

By comparing these equalities we get

$$e_i\lambda_j = -\omega_i^j(e_j)\lambda_j, \qquad \omega_j^i(e_i)\lambda_j = -\omega_j^i(e_j)\lambda_i, \tag{49}$$

and

$$\omega_j^l(e_i) = 0 \quad for \;\; l \neq i. \tag{50}$$

We now observe that $\omega_i^j(e_j) = 0$. We have $E = \lambda_1 e_1 + ... + \lambda_n e_n$ and

$$\begin{aligned}\hat{\nabla}_{e_i} E = (e_i\lambda_1)e_1 + ... + (e_i\lambda_n)e_n \\ +[\lambda_1\omega_1^i(e_i) + ... + \lambda_n\omega_n^i(e_i)]e_i.\end{aligned}$$

It follows that $e_i\lambda_j = 0$ for $i \neq j$. Using now (49) we get $\omega_i^j(e_j) = 0$. We have proved that $\hat{\nabla} e_i = 0$ for all $i = 1, ..., n$. In particular, $\hat{R} = 0$. Now, from the above formula for $\hat{\nabla} E$, we get $e_i\lambda_i = 0$. Hence all $\lambda_i$ are constant. It is now clear that $\hat{\nabla} K = 0$. □

As an immediate consequence of the results of this paper we have

**Corollary 4.7.** *Let $(g, K)$ be a statistical structure on a manifold $M$ and $\hat{\nabla} K = 0$ on $M$. Each of the following conditions implies that the metric $g$ is flat*
*1) the sectional $K$-curvature is negative*
*2) the sectional $K$-curvature has values in the interval $(0, \lambda_1^2/4)$, where $\lambda_1$ is the maximal value of $\Phi$,*
*3) $[K, K] = 0$ and $K$ is nondegenerate.*

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